Preface

The term *extremal axiom* was introduced by Carnap and Bachmann in their article *Über Extremalaxiome* (Carnap and Bachmann 1936). Axioms of this sort ascribe either maximal or minimal property to models of a theory. Examples considered by Carnap and Bachmann included: the completeness axiom in Hilbert's system of geometry, the induction axiom in arithmetic and Fraenkel's axiom of restriction in set theory. The completeness axiom (replaced later by the continuity axiom) expresses the condition of maximality of the geometric universe: that universe cannot be expanded without violating the other axioms of the system. The axiom of induction is an axiom of minimality: it expresses the idea that standard natural numbers form a minimal set satisfying the other axioms of the system. Fraenkel's axiom of restriction says that the only existing sets are those whose existence can be proved from the axioms of set theory.

There are further axioms which can be considered extremal in the above sense. Gödel's axiom of constructibility and Suszko's axiom of canonicity are minimal axioms in set theory; as in the case of Fraenkel's axiom, they also express the idea that the universe of sets should be as narrow as possible. On the other hand, axioms of the existence of large cardinal numbers are maximal axioms; they express the idea that the universe of sets should be as rich as possible.

Extremal axioms are related to the notion of an intended model. The intended model of a mathematical theory is a structure that has usually been investigated for a long time and about which we have collected essential knowledge supported also by suitable mathematical intuitions. As examples of such structures one can take for example, natural, rational or real numbers, the universe of Euclidean geometry and perhaps also the universe of all sets considered in Cantorian set theory. Given such a structure one tries to build a theory of it, ultimately an axiomatic one. It may happen that one can prove that a theory in question characterizes the intended model in a unique way (up to isomorphism or elementary equivalence). On the other hand, such theories as group theory or general topology do not have one intended model; they are thought of as theories concerning a wide class of structures.

We feel obliged to explain the reasons for collecting the material that follows within a single monograph. There are numerous publications devoted to particular problems mentioned above. However, a synthetic approach to them seems to be rare. Besides the original paper by Carnap and Bachmann, there exist only relatively few works devoted to the extremal axioms in general. Those worth mentioning include: Awodey and Reck 2002a, 2002b, Hintikka 1986, 1991, Schiemer 2010a, 2010b, 2012, 2013, Schiemer and Reck 2013, Schiemer, Zach and Reck 2015, Tarski 1940.

In our opinion, the following topics are relevant with respect to the issue of extremal axioms:

1. Revolutionary changes in mathematics of the 19th century. This concerns above all:
    (a) The rise of modern algebra understood as the investigation of arbitrary structures (domains with operations and relations) rather than – as before – looking for solutions of algebraic equations.
    (b) Discovery of systems of geometry different from the Euclidean geometry known from Euclid's *Elements* and hitherto considered the *true* system of geometry. The proposal of thinking about geometries as determined by invariants of transformations.
2. The rise of mathematical logic. The codification of the languages of logic (type theory, first-order logic, second-order logic, etc.) made it possible to talk about mathematical structures in a precise way.
3. Attempts at axiomatic characterization of fundamental types of mathematical systems. The axiomatic method previously (before the 19th century) present only in Euclid's system of geometry has become widespread in other mathematical domains.
4. Attempts at *unique* characterization of chosen mathematical structures. In these cases, where mathematical research was focused on the properties of specific a priori intended models, the question

arose of a possibility of a unique (with respect to structural or semantic properties) characterization of such models.

5. The emergence of metalogic. About one hundred years ago investigations, and the existence of several systems of logic was admitted. As a consequence, questions naturally arose concerning the comparison of these systems, their general properties, and so on.

6. Limitative results in logic and the foundations of mathematics. Metalogical reflection very soon brought important results showing the possibilities and limitations of particular logical systems, above all concerning the famous incompleteness results. It became evident that certain methodological ideals can not be achieved simultaneously – for instance, “good” deductive properties and a great “expressive power” are in conflict, in a precisely defined sense.

7. Philosophical reflection on mathematical intuition as a major factor in the context of discovery.

Our main research goal is modest; we admit that the material covered by the book is to a large extent known to specialists. However, we believe that presenting the origins of extremal axioms and the development of research on them could be of some value to the reader who is interested in mathematical cognition. We make some use of the material contained in: Pogonowski 2011, 2016, 2017, 2018a, 2018b.

The book consists of three parts focusing in turn on logical, mathematical and cognitive aspects of extremal axioms.

Part I: Logical aspects. We begin here with a discussion concerning the relations between mathematical theories and their models. Then we present remarks about the origin of extremal axioms as well as the origin of certain metalogical properties, notably those of categoricity and completeness. We recall some famous limitative theorems which show possibilities and limitations in the unique characterization of models. The notion of the expressive power of logic is useful in this respect. The final chapter of this part gives examples of important results from model theory related to the properties of categoricity and completeness.

Part II: Mathematical aspects. Here we discuss the role of particular extremal axioms and selected properties of theories based on extremal axioms. First, we present results related to the continuity axiom in geometry, algebra and analysis. Then we recall the role played by the induction axiom in arithmetic. Finally, we discuss two types of extremal axioms in set theory, namely the axioms of restriction and the axioms of the existence of large cardinal numbers.

PART III: COGNITIVE ASPECTS. The main topic of this part is mathematical intuition. This cognitive ability should be conceived of as a major factor responsible for characterization of intended models. We discuss selected philosophical standpoints concerning this notion as well as related opinions among professional mathematicians. To the well-known contexts of discovery and justification we add a new one, namely the context of transmission, which embraces activities related to learning and teaching mathematics, as well as the popularization of mathematics.

The book is addressed mainly to cognitive scientists interested in mathematical cognition. We are aware that logicians and mathematicians are well familiar with extremal axioms and their consequences in mathematics itself. Cognitive scientists, in turn, could be interested in the ways of characterization of models of mathematical theories or, more generally, in cognitive access to fragments of mathematical reality described by the theories in question.

The mathematical and logical terminology used in the book is standard (however, there are some differences in the notations used by the authors whose works are discussed). We do not explicate the elementary formal concepts, but we provide definitions of those advanced ones which are important for the main subject. It is assumed that the readers are acquainted with the fundamentals of formal logic and the rudiments of set theory (including the notions of ordinal and cardinal numbers) as well as elementary algebra (including such notions as isomorphism, group, and field) and a little of general topology. The presentation is in rather simple English due to the fact that the author is not a native speaker of the language.

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